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The representation of the Heisenberg–Euler Lagrangian by means of special functions

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Abstract. A representation of the real part of the Heisenberg–Euler Lagrangian density in quantum electrodynamics by means of special functions is obtained. It is shown that this representation is very convenient for numerical calculations of the real part of the Heisenberg–Euler Lagrangian density. It is indicated that this representation is of use for calculations of a quantum electrodynamical field energy density in the absence of real charges and for calculations of polarisation and magnetisation of the vacuum.

1. Introduction

In quantum electrodynamics (QED) the effective one-loop Lagrangian (Heisenberg and Euler 1936, Weisskopf 1936, Schwinger 1951), usually called the Heisenberg–Euler Lagrangian, describes the effective nonlinear interaction of the electromagnetic fields due to a single closed electron loop. The Heisenberg–Euler Lagrangian enables us to describe quantitatively such electromagnetic processes as the propagation of a photon in a magnetised vacuum, photon splitting in the presence of a strong magnetic field (Białynicka-Birula and Białynicki-Birula 1970, Adler *et al* 1970, Adler 1971, Stoneham 1979) and scattering of light by light in a vacuum (Euler 1936).

When the external field is pure magnetic, the Heisenberg–Euler Lagrangian density was expressed by means of the generalised gamma function and elementary functions (Dittrich *et al* 1979). Numerical values of the Lagrangian density were given by Valluri *et al* (1982). A similar representation in a pure electric field case was also found (Dittrich *et al* 1979, Valluri *et al* 1982).

The main purpose of this paper is to generalise the results mentioned above in the case when both electric and magnetic fields coexist. The real part of the Heisenberg–Euler Lagrangian will be expressed by means of the following mathematical functions: the cosine integral $\mathrm{Ci}(u)$, the sine integral $\mathrm{Si}(u)$, the exponential integral $\mathrm{Ei}(u)$ and elementary functions.

We shall show that the representation obtained is very convenient to calculate the numerical values of the real part of the effective Lagrangian density.

The paper is organised as follows: in § 2 we shall present our representation of the real part of the effective Heisenberg–Euler Lagrangian density. The main result of this paper is given by equation (2.11). In § 3 we shall indicate possible physical applications of our representation.

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2. Representation of the Lagrangian density

Heisenberg and Euler (1936), Weisskopf (1936) and Schwinger (1951) obtained the following form of the effective Lagrangian density $\mathcal{L}_{\text{eff}}(\boldsymbol{E}, \boldsymbol{B})$

$$\mathcal{L}_{\text{eff}}(\boldsymbol{E}, \boldsymbol{B}) = \mathcal{S} - \frac{1}{8\pi^2}\int_C \frac{\mathrm{d}u}{u^3}\exp(-m^2u)\left((eu)^2\mathcal{P}\frac{\operatorname{Re}[\cosh(euW)]}{\operatorname{Im}[\cosh(euW)]} - 1 + \tfrac{2}{3}(eu)^2\mathcal{S}\right) \tag{2.1}$$

where

$$W = \sqrt{2(-\mathcal{S} + \mathrm{i}\mathcal{P})} \tag{2.2}$$

e and m denote the charge and the mass of the electron (we adopt the usual convention $\hbar = c = 1$). The field invariants $\mathcal{S}$ and $\mathcal{P}$ are defined by

$$\mathcal{S} = \tfrac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) \tag{2.3}$$

$$\mathcal{P} = \boldsymbol{E}\cdot\boldsymbol{B} \tag{2.4}$$

where $\boldsymbol{B}$ and $\boldsymbol{E}$ denote the magnetic induction and electric field respectively.

From the equation (Weisskopf 1936)

$$(eu)^2\mathcal{P}\frac{\operatorname{Re}[\cosh(euW)]}{\operatorname{Im}[\cosh(euW)]} - 1 + \tfrac{2}{3}(eu)^2\mathcal{S} = \tilde{x}\tilde{y}u^2\coth(\tilde{x}u)\cot(\tilde{y}u) - 1 - \tfrac{1}{3}(\tilde{x}^2 - \tilde{y}^2)u^2 \tag{2.5}$$

where

$$\tilde{x} = e[-\mathcal{S} + (\mathcal{S}^2 + \mathcal{P}^2)^{1/2}]^{1/2} \tag{2.6}$$

$$\tilde{y} = e[\mathcal{S} + (\mathcal{S}^2 + \mathcal{P}^2)^{1/2}]^{1/2} \tag{2.7}$$

one can show that the integrand of the expression given by the right-hand side of equation (2.1) has poles at the points

$$u_k = \frac{k\pi}{e[\mathcal{S} + (\mathcal{S}^2 + \mathcal{P}^2)^{1/2}]^{1/2}} \qquad k = 1, 2, \ldots$$

The integration in equation (2.1) is carried out over the contour C bypassing the poles as shown in figure 1.

It is obvious that the right-hand side of equation (2.1) is not convenient for numerical calculations. To overcome this difficulty we transform the right-hand side of equation (2.5) (compare Claudson *et al* 1980)

$$\begin{aligned}\tilde{x}\tilde{y}u^2\coth(\tilde{x}u)\cot(\tilde{y}u) - 1 - \tfrac{1}{3}(\tilde{x}^2 - \tilde{y}^2)u^2 &= -\frac{2\tilde{x}\tilde{y}^3u^4}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{\tilde{y}^2u^2 + k^2\pi^2}\coth\left(\frac{\tilde{x}}{\tilde{y}}k\pi\right)\\ &\quad + \frac{2\tilde{x}\tilde{y}^3u^4}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{\tilde{y}^2u^2 - k^2\pi^2}\coth\left(\frac{\tilde{x}}{\tilde{y}}k\pi\right)\end{aligned}$$

+ the expression obtained from the first component by the replacement rule

$$\begin{Bmatrix}\tilde{x} \to -\mathrm{i}\tilde{y}\\ \tilde{y} \to -\mathrm{i}\tilde{x}\end{Bmatrix}. \tag{2.8}$$

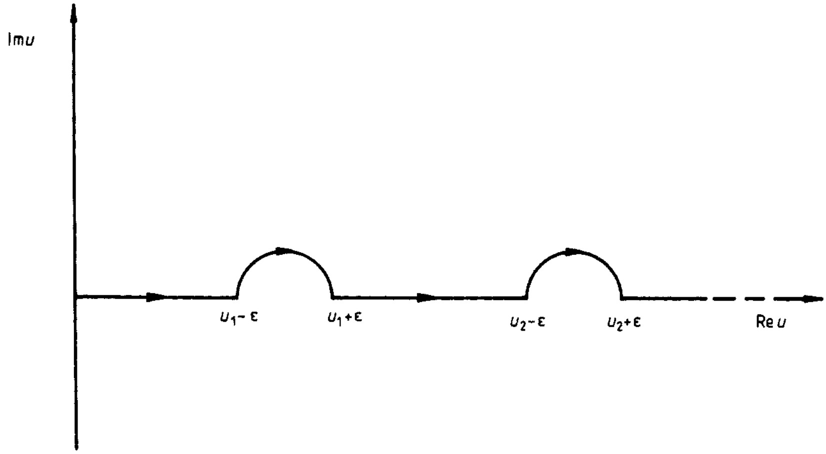

Figure 1. The curve of the integration C of the expression given by the right-hand side of equation (2.1). The curve C bypasses the points

$$u_k = \frac{k\pi}{e[\mathcal{S}+(\mathcal{S}^2+\mathcal{P}^2)^{1/2}]^{1/2}} \qquad k=1,2,\ldots$$

The symbol u denotes the complex variable of the integration. Re u denotes its real part and Im u is its imaginary part. ε denotes the small positive parameter. To calculate the value of the integral given by the right-hand side of equation (2.1) the limit $\varepsilon \to 0$ should be taken.

Next we insert equation (2.8) into equation (2.1) and apply the integral formulas (Gradshteyn and Rhyzhik 1980)

$$\int_0^\infty \frac{t\exp(-\mu t)}{t^2+1}\,\mathrm{d}t = -\mathrm{Ci}(\mu)\cos\mu - \mathrm{si}(\mu)\sin\mu \tag{2.9}$$

for

$$\mu = k\pi m^2/\tilde{x} \qquad (k=1,2,\ldots)$$

and

$$\mathcal{P}\int_0^\infty \frac{t\exp(-\tilde{\mu}t)}{t^2-1}\,\mathrm{d}t - \exp(\tilde{\mu}\ \mathrm{Ei}(-\tilde{\mu}) - \exp(-\tilde{\mu})\,\mathrm{Ei}(\tilde{\mu}) \tag{2.10}$$

for

$$\tilde{\mu} = k\pi m^2/\tilde{y} \qquad (k=1,2,\ldots)$$

(the symbol $\mathcal{P}$ in equation (2.10 denotes the principal value of the integral).

Finally we arrive at the following expression for the real part of the effective Lagrangian

$$\mathrm{Re}\,\mathcal{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B}) = \mathcal{S} - \frac{\alpha}{\pi^2}|\mathcal{P}|\sum_{k=1}^{\infty} a_k + d_k \tag{2.11}$$

where

$$a_k = k^{-1}\coth(yk\pi/x)[\mathrm{Ci}(k/x)\cos(k/x) + \mathrm{Si}(k/x)\sin(k/x)] \tag{2.12}$$

$$d_k = k^{-1}\coth(xk\pi/y)[\exp(k/y)\,\mathrm{Ei}(-k/y) - \exp(-k/y)\,\mathrm{Ei}(k/y)] \tag{2.13}$$

and dimensionless quantities x and y are defined by

$$x = \tilde{x}/\pi m^2 \tag{2.14}$$

$$y = \tilde{y}/\pi m^2 \tag{2.15}$$

α denotes the fine structure constant. (The symbol $\mathscr{P}$ in equation (2.10) denotes the principal value of the integral.)

The cosine integral $\mathrm{Ci}(u)$ is defined by

$$\mathrm{Ci}(u) = -\int_u^\infty \frac{\cos t}{t}\,\mathrm{d}t \qquad \text{for } u > 0 \tag{2.16}$$

(Korn and Korn 1961); the sine integral $\mathrm{Si}(u)$ is given by

$$\mathrm{Si}(u) = -\int_u^\infty \frac{\sin t}{t}\,\mathrm{d}t \qquad \text{for } u > 0. \tag{2.17}$$

The exponential integral $\mathrm{Ei}(u)$ is defined by

$$\mathrm{Ei}(u) = \begin{cases} -\displaystyle\int_{-u}^\infty \frac{\exp(-t)}{t}\,\mathrm{d}t & \text{for } u < 0 \\ -\displaystyle\lim_{\varepsilon\to 0+}\left(\int_{-u}^{-\varepsilon} \frac{\exp(-t)}{t}\,\mathrm{d}t + \int_{\varepsilon}^\infty \frac{\exp(-t)}{t}\,\mathrm{d}t\right) & \text{for } u > 0. \end{cases} \tag{2.18}$$

We adopt the notation of Korn and Korn (1961) to denote the special functions mentioned above. The numerical values of these functions were given by Etherington (1958), Harris (1957), Miller and Hurst (1958), Korn and Korn (1961) and Abramowitz and Stegun (1970).

Equation (2.11) presents the main result of this paper. To the best of our knowledge the formula (2.11) is unknown in literature. However, this equation can be obtained from somewhat similar results of Claudson *et al* (1980) if one specifies a general gauge group to the QED case and applies our equations (2.9) and (2.10).

One can show that for an arbitrary number Δ satisfying the condition

$$0 < \Delta < 1$$

the sequence $a_k + d_k$ has the following behaviour for large k

$$|a_k + d_k| \leqslant G/k^{2-\Delta} \tag{2.19}$$

where the constant G does not depend on k. This behaviour guarantees convergence of the series $\Sigma_{k=1}^{\infty} a_k + d_k$.

3. Applications of equation (2.11)

In a similar way one can calculate the QED energy density $\mathscr{H}$ of the electromagnetic field in the absence of real charges. The energy density $\mathscr{H}$ of such a system is given by the relation

$$\mathscr{H} = \boldsymbol{E} \cdot \boldsymbol{D} - \mathrm{Re}\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E}, \boldsymbol{B}) \tag{3.1}$$

where the electric displacement $\boldsymbol{D}$ is defined by the relation

$$\boldsymbol{D}=\frac{\partial\,\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})}{\partial\boldsymbol{E}}=\frac{\partial\,\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})}{\partial\mathscr{S}}\boldsymbol{E}+\frac{\partial\,\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})}{\partial\mathscr{P}}\boldsymbol{B}. \tag{3.2}$$

Equations (3.1) and (3.2) give

$$\mathscr{H}=\frac{\partial\,\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})}{\partial\mathscr{S}}\boldsymbol{E}^2+\frac{\partial\,\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})}{\partial\mathscr{P}}\mathscr{P}-\mathrm{Re}\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B}). \tag{3.3}$$

Equation (3.3) can be rewritten in the form

$$\mathscr{H}=\tfrac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)+\frac{\partial(\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})-\mathscr{S})}{\partial\mathscr{S}}\boldsymbol{E}^2+\frac{\partial(\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})-\mathscr{S})}{\partial\mathscr{P}}$$
$$-\mathscr{P}-(\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})-\mathscr{S}). \tag{3.4}$$

The first component on the right-hand side of equation (3.4) represents the Maxwellian energy density of the electromagnetic field. The rest is due to the interaction of the field with vacuum fluctuations.

The derivatives

$$\frac{\partial(\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})-\mathscr{S}/4\pi)}{\partial\mathscr{S}}\qquad\text{and}\qquad\frac{\partial(\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})-\mathscr{S}/4\pi)}{\partial\mathscr{P}}$$

can be easily expressed in terms of the sine integral, the cosine integral and the exponential integral by differentiating equation (2.11) with respect to the field invariants $\mathscr{S}$ and $\mathscr{P}$. Therefore the energy density $\mathscr{H}$ can be represented by means of these special functions as well.

By a similar method we are able to calculate the polarisation density $\boldsymbol{P}$ and the magnetisation density $\boldsymbol{M}$ created by the external electromagnetic field

$$\boldsymbol{P}=\boldsymbol{D}-\boldsymbol{E} \tag{3.5}$$

$$\boldsymbol{U}=\boldsymbol{B}-\boldsymbol{H} \tag{3.6}$$

where the magnetic field $\boldsymbol{H}$ is defined by

$$\boldsymbol{H}=-\frac{\partial\,\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})}{\partial\boldsymbol{B}}=\frac{\partial\,\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})}{\partial\mathscr{S}}\boldsymbol{B}-\frac{\partial\,\mathrm{Re}\,\mathscr{L}_{\mathrm{eff}}(\boldsymbol{E},\boldsymbol{B})}{\partial\mathscr{P}}\boldsymbol{E}.$$

Numerical values of the QLD energy density $\mathscr{H}$, the polarisation density $\boldsymbol{P}$ and the magnetisation density $\boldsymbol{U}$ wel be given in a subsequent paper.

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